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Abstract
Iterated function systems (IFS) are interesting parametric models for generating
fractal sets and functions. The general idea is to compress, deform and translate
a given set or function with a collection of operators and to iterate the procedure.
Under weak assumptions, IFS possess a unique fixed point which is in general
fractal. IFS were introduced in a deterministic context, then were generalized
to the random setting on abstract spaces in the early 1980 s. Their adaptation
to random signals was carried out by Hutchinson and Rüschendorff [9] by
considering random operators. This study extends their model with not only
random operators but also a random underlying construction tree. We show that
the corresponding IFS converges under certain hypothesis to a unique fractal
fixed point. Properties of the fixed point are also described.

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1. Introduction

Signals presenting scale invariance have been widely studied during the past 20 years. The
name scale invariance refers to signals not presenting any characteristic scale, each scale
playing a similar role. Applications of such signals are wide and range from biology [1] to
finance [2], and from network traffic [3] to turbulence [4]. Iterated function systems (IFS) have
received interest in image compression and decompression, where attempts are made to solve
the IFS inverse problem: identifying the parameters of an IFS whose attractor is a target image
[5]. IFS can also be used for fractal interpolation [6, 7]. The classical IFS considered in the
literature are deterministic IFS, where the object (a set, a measure, a function) is transformed
by means of deterministic operators. The formalism was first introduced on abstract sets, then
adapted to produce fractal measures and functions. Signals obtained from this procedure can
be multifractals [8]. Generalization to the random setting has previously been carried out by
Hutchinson and Rüschendorff [9], where only IFS operators are random. This study extends

Figure 1. Shadow of a shape drawn in the sand of a Queensland beach at dusk. Credit G Decrouez.

their model by allowing more randomness. As an example of a natural shape that may be modelled as the attractor of a random IFS, figure 1 shows the shadow of a shape drawn in the sand of a Queensland beach.

IFS were first introduced over the space of compact subsets of $\mathbb{R}^2$, usually denoted by $\mathcal{H}(\mathbb{R}^2)$. This space is particularly interesting when dealing with black and white pictures. $\mathcal{H}(\mathbb{R}^2)$ is generally endowed with the Hausdorff metric $d_{\mathcal{H}}$. For A and B in $\mathcal{H}(\mathbb{R}^2)$,

$$d_{\mathcal{H}}(A, B) = \max[d(A, B), d(B, A)] \tag{1}$$

where $d(A, B) = \max[d(x, B), x \in A]$ and $d(x, A) = \min[d(x, y), y \in B]$ for any $x \in A$.

Let $\omega : \mathbb{R}^2 \to \mathbb{R}^2$ be a contractive application with the contraction ratio s. Then $\omega : \mathcal{H}(\mathbb{R}^2) \to \mathcal{H}(\mathbb{R}^2)$ defined by

$$\omega(B) = \{\omega(x) | x \in B\} \, \forall B \in \mathcal{H}(\mathbb{R}^2) \tag{2}$$

is contractive in the metric space $(\mathcal{H}(\mathbb{R}^2), d_{\mathcal{H}})$ with the contraction constant s. Now consider a set of M contractive maps ω_n with contraction ratios $s_n, n = 1, \ldots, M$. Then $W : \mathcal{H}(\mathbb{R}^2) \to \mathcal{H}(\mathbb{R}^2)$

$$W(B) = \bigcup_{i=1}^{M} \omega_i(B) \tag{3}$$

is contractive in $(\mathcal{H}(\mathbb{R}^2), d_{\mathcal{H}})$ with the contraction factor $s = \max_n s_n$ [10]. In other words, the operator W starts with an initial set B and compute its image by taking the union of contracted and translated copies of the original set B. By completeness of the metric space $(\mathcal{H}(\mathbb{R}^2), d_{\mathcal{H}})$, it follows from the Banach fixed point theorem that the IFS possesses a unique fixed point B^*, which satisfies $W(B^*) = B^*$. Many well-known fractal sets such as the Sierpinski gasket are obtained from this procedure. Such attractors can also produce images close to shapes found in nature, such as the famous example of fern leaves [10]. One can associate a tree with this construction, which is deterministic in the present setting. For IFS with M maps, the underlying construction tree is an M-ary tree, where each node possesses exactly M offspring, as illustrated in figure 2, for $M = 2$.
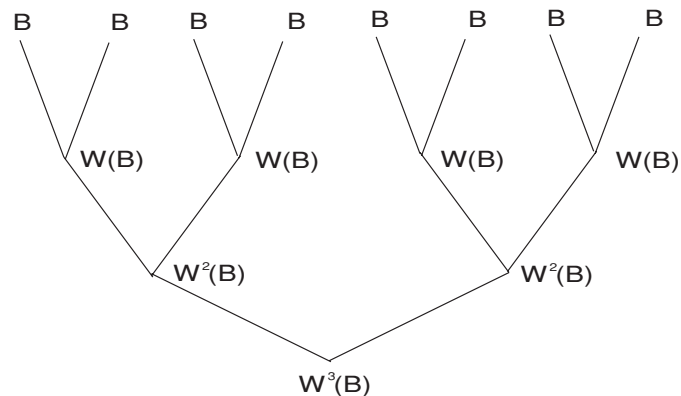
Figure 2. Underlying binary tree associated with a deterministic IFS ($M = 2$). $W^i(B)$ for $i = 2, 3$ represents the ith iterate of W. Under mild conditions, convergence to a fixed point occurs when the tree considered is infinite. The fixed point is then at the bottom of the tree (root).

This formalism was adapted to produce fractal measures and signals, first in a deterministic setting. Random IFS were introduced in the 1980s on abstract mathematical sets [11, 12]. More recently, Hutchinson and Rüschendorff have randomized the construction for signals, where operators are randomized, but the deterministic tree structure is retained. In this study we further randomize this model, allowing a random construction tree, or random branching process. Applications of branching processes, which started with the study of demography of populations [13], have been applied to many areas of science and provide good models in biology [14]. A good review of branching processes can be found in [15]. Random cascades and measures defined on the boundary of random trees have also been widely studied. See, for example, the works of Peyrière [16], Hawkes [17], Burd and Waymire [18], Liu [19, 20], Mörters and Shieh [21]. Their works differ from the present study as they consider measures defined as cascades over Galton–Watson trees, whereas we obtain functions defined over compact intervals. However, it is important to note that the two constructions are not completely disjoint, as fractal measures obtained from a cascade can have full support, and in that case define a self-similar function. In general, cascades differ from IFS in that they can have fractal support. However, IFS allow more general fixed points than cascades, since they are not restricted to functions being increasing.

This study is based on the previous idea of iteration on a Galton–Watson tree. Its novelty is to introduce new models for generating fractal signals using branching processes.

This paper is organized as follows. We first introduce the complete metric space of random signals, where the fixed point of the random IFS lies. We then build the probability space of extended Galton–Watson trees, presented in section 3. Extended trees are Galton–Watson trees whose branches are endowed with a random operator. In section 4, we derive precise conditions under which the IFS possess a unique fixed point and illustrate the type of signals one can obtain with this new model. In the last section we study various properties of the fixed point, such as conditions for it to be continuous. Furthermore, it is shown in [10] that the fractal attractor of a deterministic IFS continuously depends on the parameters of the IFS. We extend this result and show here, in a special case, that the moments of the fixed point continuously depend on the probability vector of the random variable giving the number of maps used at each iteration of the algorithm. Finally, we give empirical results on the multifractal behaviour on the fixed point.

2. Iterated function systems on functions

In this section, we present the model and introduce the working spaces. The random IFS model presented in part B is referred to as a Galton–Watson IFS, referring to the random structure of its underlying construction tree.

2.1. Deterministic IFS

Let $L_p(\mathbb{X})$ be the space of p-integrable signals $\mathbb{X} \to \mathbb{R}$ where $\mathbb{X}$ is a compact subset of the real line. $\|\cdot\|_p$ is the usual norm defined on $L_p(\mathbb{X})$: $\|f\|_p = (\int |f|^p \, d\mu)^{1/p}$ where μ is the Lebesgue measure, leading to the natural metric d_p defined by $d_p(f, g) = \|f - g\|_p$ where f and g are in L_p. It is common to consider without loss of generality the case $\mathbb{X} = [0, 1]$.

An IFS consists of recursively applying an operator T with certain properties. Starting with an initial function f_0, we denote by $T^n f_0$ the nth iterate of T acting on f_0. For a class of operators T, the IFS converges to a function f^*:

$$T^n f_0 \to f^* \qquad \text{as} \quad n \to +\infty \tag{4}$$

in $L_p(\mathbb{X})$. f^* is the unique function satisfying $f = Tf$. That is, f^* is the fixed point or attractor of the IFS associated with T. It is generally assumed that T can be decomposed into a set of M nonlinear operators $\phi_i : \mathbb{R} \times \mathbb{X} \to \mathbb{R}$ for $1 \leqslant i \leqslant M$. Each ϕ_i deforms the original signal and maps it to a subinterval $\mathbb{X}_i = \varrho_i(\mathbb{X})$ of $\mathbb{X}$. Specifically,

$$(Tf)(x) = \sum_{i=1}^{M} \phi_i\big[f\big(\varrho_i^{-1}(x)\big), \varrho_i^{-1}(x))\big]\mathbf{1}_{\varrho_i(\mathbb{X})}(x) \tag{5}$$

where the $\{\varrho_i(\mathbb{X})\}_{i=1}^{M}$ are affine maps and partition $\mathbb{X}$ into M subintervals. $\mathbf{1}_{\varrho_i(\mathbb{X})}$ is the indicator function of the interval $\varrho_i(\mathbb{X})$. In (5), ϕ_i are functions of two variables. The second variable is however optional and one can define the operator T with $\phi_i : \mathbb{R} \to \mathbb{R}$. The underlying construction tree is an M-ary deterministic tree. Conditions of convergence of the IFS are derived explicitly in [9] for nonlinear functions $\phi_i : \mathbb{R} \to \mathbb{R}$. The result can easily be generalized to functions $\phi_i : \mathbb{R} \times \mathbb{X} \to \mathbb{R}$, as above.

Theorem 2.1. *Suppose that ϱ_i are strict contractions with contraction factors $r_i < 1$ for $i = 1, \dots, M$, and that ϕ_i are Lipschitz in their first variable with Lipschitz constants S_i, i.e. $\forall (u_1, u_2, v) \in \mathbb{R}^2 \times \mathbb{X} |\phi_i[u_1, v] - \phi_i[u_2, v]| \leqslant S_i|u_1 - u_2|$. If for some p, $\lambda_p = \sum_{i=1}^{M} r_i S_i^p < 1$ and $\sum_{i=1}^{M} r_i \int |\phi_i(0, x)|^p \, dx < \infty$, then T has a unique fixed point in $L_p(\mathbb{X})$.*

This is a specific case of theorem 4.1, so the proof is not given here. The conditions for convergence are quite weak. The second condition only requires that ϕ_i must be p integrable with respect to their second variable.

Figure 3 presents attractors of two different IFS, one continuous and one discontinuous. Conditions for continuity are derived in section 5.1 in a more general setting.

The deterministic model acting on functions is not flexible enough to model natural signals. This is mainly due to its deterministic self-similarity as observed in figure 3. One way to break this pattern is to add randomness to the construction. Therefore section 2.3 defines random IFS with random operators and a random construction tree.

2.2. $\mathbb{L}_p$ spaces

Before giving the definition of a Galton–Watson IFS, we need to specify the space where the fixed point lies. Let $(\Sigma, \mathcal{F}, P)$ be a probability space, then a p-integrable random process is a
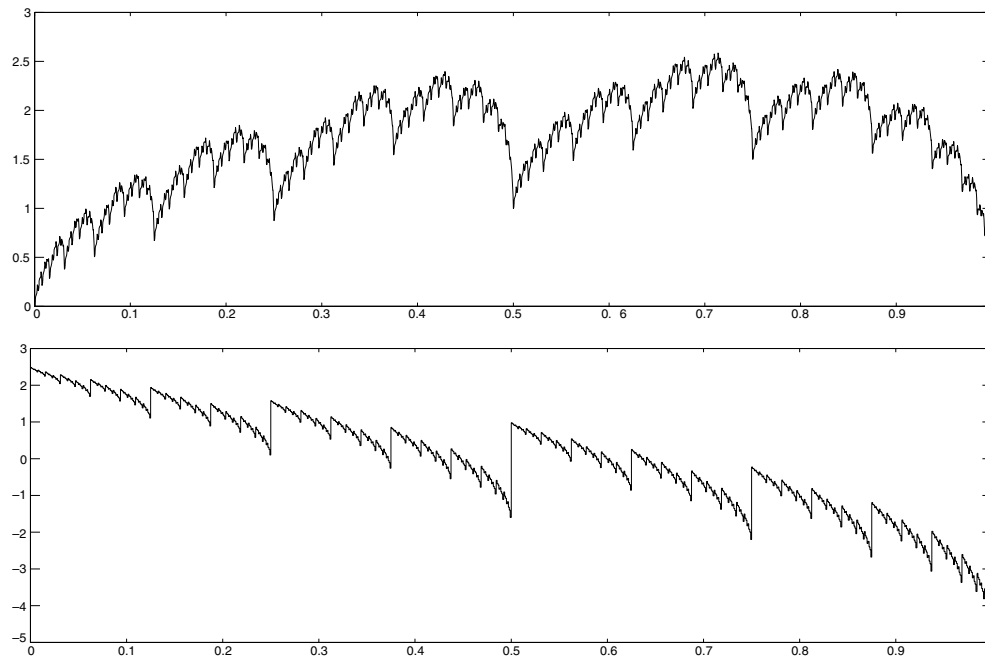
Figure 3. Top signal: continuous attractor of the IFS defined with the maps $\phi_1(u, v) = s_1 u + v^3$ and $\phi_2(u, v) = s_2 u + (1 - v^2)$, where $s_1 = s_2 = 0.75$. The bottom discontinuous signal is also obtained as the fixed point of an IFS, whose parameters are $\phi_1(u, v) = s_1 u + 1$ and $\phi_2(u, v) = s_2 u - 1$, where $s_1 = 0.6$ and $s_2 = 0.8$. $\mathbb{X} = [0, 1]$ in both cases.

random variable $f : \Sigma \to L_p(\mathbb{X})$. Define

$$\mathbb{L}_p = \left\{ f : \Sigma \to L_p(\mathbb{X}) \mid \mathbb{E}\left[\|f\|_p^p\right] < +\infty \right\}$$

where $\mathbb{E}$ denotes expectation under P. We denote by f_σ a realization of the random process $f \in \mathbb{L}_p$, where $\sigma \in \Sigma$, which will be useful in the proof of theorem 4.1. $f(x) : \Sigma \to \mathbb{R}$ is the random variable obtained by evaluating f at x. The norm of $f \in \mathbb{L}_p$, $p \geqslant 1$, is defined as

$$\|f\|_p^* = \mathbb{E}^{\frac{1}{p}}\left[\|f\|_p^p\right]. \tag{6}$$

The metric d_p^* is then defined as follows: $d_p^*(f, g) = \|f - g\|_p^*$. It is straightforward to adapt the proof of the Riesz–Fisher theorem [29] to show that $(\mathbb{L}_p, d_p^*)$ is a Banach space.

2.3. Galton–Watson IFS

The operator T acting over the space $\mathbb{L}_p$ is now defined as follows:

$$(Tf)(x) = \sum_{j=1}^{\nu} \phi_j\left[f^{(j)}\left(\varrho_j^{-1}(x)\right), \varrho_j^{-1}(x)\right]\mathbf{1}_{\varrho_j(\mathbb{X})}(x) \tag{7}$$

where $(\nu, \phi_1, \varrho_1, \ldots, \phi_\nu, \varrho_\nu)$ is random and $f^{(j)}$ are iid copies of f. The ϱ_j are affine maps and randomly partition $\mathbb{X}$ into ν subintervals. The contraction factor of ϱ_j is the random variable r_j, such that $0 < r_j < 1$ almost surely. ϕ_j are functions of two variables, Lipschitz in their first variable, with a random Lipschitz factor S_j. ν is distributed according to a probability vector $\mathbf{q} = (q_1, q_2, \ldots)$. The underlying construction tree has therefore a random

number of offspring at each node. Assuming that, in this construction, the random variable ν is independent and identically distributed from one node to another, the construction tree is of Galton–Watson type [22], hence the name of the IFS.

3. Space of extended trees

An ad hoc structure of Σ is needed in order to build iid copies of the signal f. We show how to do this in the present section using extended Galton–Watson trees.

The construction of the probability space of extended Galton–Watson trees relies on two famous theorems in measure theory: the Ionescu-Tulcea theorem and the Daniell–Kolmogorov extension theorem. We use the first theorem to build a probability space of the first n generations of extended trees for any finite integer n, then extend the construction to infinite trees using the Daniell–Kolmogorov extension theorem. An element of that space therefore consists of a realization of a Galton–Watson tree whose branches are equipped with realizations of the IFS operators.

Ionescu-Tulcea [24]. The result of Ionescu-Tulcea relies on the concept of probability kernels. Let $(A_1, \mathcal{A}_1)$ and $(A_2, \mathcal{A}_2)$ be two measurable spaces. A probability kernel is a function $\kappa_2 : A_1 \times \mathcal{A}_2 \to [0, 1]$ such that for all $E \in \mathcal{A}_2, a \mapsto \kappa_2(a, E)$ is a measurable function on A_1 and such that for all $a \in A_1, E \mapsto \kappa_2(a, E)$ is a probability measure on $(A_2, \mathcal{A}_2)$. We interpret κ_2 as a probability distribution on $(A_2, \mathcal{A}_2)$ conditioned on state $a \in A_1$ and write it either $\kappa_2(E|a)$ or $\kappa_2(a, E)$. Let $\kappa_3 : A_1 \times A_2 \times \mathcal{A}_3 \to [0, 1]$ be a probability measure on $(A_3, \mathcal{A}_3)$ given we were in state (a_1, a_2) for $a_i \in A_i, i = 1, 2$, in the previous step. Then the kernel $\kappa_2 \otimes \kappa_3$ defined as

$$(\kappa_2 \otimes \kappa_3)(a_1, E) = \iint \mathbf{1}_E(b, c)\kappa_2(a_1, \mathrm{d}b)\kappa_3(a_1, b, \mathrm{d}c) \qquad (8)$$

measures Borel subsets E of $\mathcal{A}_2 \times \mathcal{A}_3$ from an initial state $a_1 \in A_1$. Ionescu-Tulcea lets us chain correctly n measurable spaces $(A_i, \mathcal{A}_i), i = 1, \ldots, n$, by defining a joint probability on the product space $\prod_{i=1}^n A_i$ from n probability kernels κ_i. The result of Ionescu-Tulcea is then the following [24]. Let κ_1 be a probability measure on $(A_1, \mathcal{A}_1)$ and for all $n \geqslant 2, \kappa_n : \left(\prod_{i=1}^{n-1} A_k\right) \times \mathcal{A}_n \to [0, 1]$ a probability kernel. Then there exists a unique probability measure on $\prod_{i=1}^n A_k$ given by $\bigotimes_{i=1}^n \kappa_i$, a generalization of equation (8).

Daniell–Kolmogorov [30]. The Daniell–Kolmogorov extension theorem extends a measure defined on a sequence of finite product spaces to a measure on an infinite product space. Let $A_1, A_2, \ldots$, be a sequence of measurable spaces and μ_n a measure on the product space $A_1 \times \cdots \times A_n$. We say that the sequence of probability measures μ_n forms a projective family if $\mu_{n+1}(\cdot \times A_{n+1}) = \mu_n$ for all $n \in \mathbb{N}$. Daniell–Kolmogorov state that if μ_n forms a projective family, then there exists a measure μ on $\prod_{i=1}^\infty A_i$ such that μ_n is equal to the projection of μ onto $\prod_{i=1}^n A_i$.

Space of extended trees. Let $(\Delta, \mathcal{D}, P)$ be the probability space of elements of the form

$$\delta = (\nu, \phi_1, \varrho_1, \ldots, \phi_\nu, \varrho_\nu).$$

An element of this space carries information about the node to which it is attached: it contains the number of children of the node (random variable ν) and the operators attached to each of its branches. The probability measure $\kappa_1 = P$ lets us build the sample space for first generation of the tree, denoted by $K^1 = \Delta$. We define K^j the sample space of the jth generation of the tree by

$$K^j := \{\{\delta(i)\} \mid i = 1, \ldots, Z_j\, \delta(i) \in \Delta\, Z_j \in \{1, 2, 3, \ldots\}\}.$$
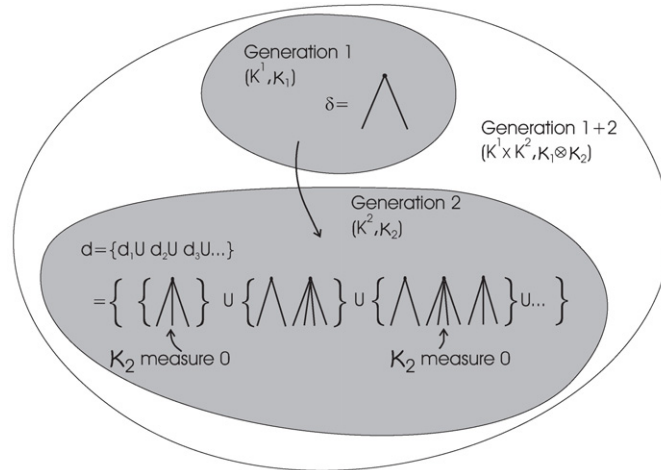
Figure 4. Spaces K^1, K^2 and $K^1 \times K^2$ with their respective probability measures κ_1, κ_2 and $\kappa_1 \otimes \kappa_2$. $\delta \in K^1$ has two children. Conditionally on δ, only $d_2 \in K^2$ represented here has non-zero measure as it is the only element composed of two families. κ_2 assigns measure to each family in d_2 independently. To keep the figure simple, operators attached to the branches of the tree are not represented.

The σ-algebra associated with K^j is

$$\mathcal{D}_j = \sigma\left(\bigcup_{k \geqslant 1} \mathcal{D}^k \right) = \{d_1 \cup d_2 \cup \cdots \mid d_i \in \mathcal{D}^i\}, \tag{9}$$

where $\mathcal{D}^k = \mathcal{D} \times \cdots \times \mathcal{D}\, k$ times. In (9), note that the right-hand side does not depend on j. This comes from the definition of K^j which is the same for all $j \geqslant 2$. The σ-algebra attached to each K^j is therefore the same. Also, we need to consider the smallest σ-algebra spanned by the union of $\mathcal{D}^k$ since the union of σ-algebras is not in general a σ-algebra.

The construction of κ_2 supposes we know the first generation and in particular its size Z_1. For $d = d_1 \cup d_2 \cup \cdots \in \mathcal{D}_2$, with $d_j = E_1^j \times \cdots \times E_j^j \in \mathcal{D}^j$, we define

$$\kappa_2(d|Z_1) = \prod_{i=1}^{Z_1} P\big(E_i^{Z_1}\big). \tag{10}$$

Sets d_i for $i \neq Z_1$ therefore receive a zero measure. This is illustrated in figure 4. By taking the product of $P\big(E_i^{Z_1}\big)$ we ensure independence from one node of the tree to the next.

The procedure for constructing κ_2 is repeated n times to build a probability measure on the first n generations $\prod_{i=1}^n K^i$, from Ionescu–Tulcea. Then, Daniell–Kolmogorov let us extend the measure to infinite trees since by construction $\bigotimes_{i=1}^n \kappa_i$ forms a projective family. Let K be the infinite product space, $\mathcal{K}$ its σ-algebra and κ the probability distribution over this space.

Definition 3.1. $(K, \mathcal{K}, \kappa)$ *is the probability space of extended Galton–Watson trees.*

By construction, extended trees are Galton–Watson trees whose branches are marked with random operators. We use classical notation to label nodes and branches of the tree: let $\emptyset$ be the root of the tree and $\nu_\emptyset$ be the number of branches rooted at $\emptyset$. Then each node coming from the root is denoted by i, for $i = 1, \ldots, \nu_\emptyset$. The second generation of the tree is denoted

by ij for $1 \leqslant j \leqslant \nu_i$. More generally, a node is an element of $U = \bigcup_{n \geqslant 0} \mathbb{N}^{*n}$ and a branch is a couple of nodes (u, uj) where $u \in U$ and j is a strictly positive integer. Lastly, we consider $T_u(k)$ the subtree of $k \in K$ rooted at u: $T_u(k) = \{v \mid v \in U \text{ and } uv \in k\}$. By construction, the random variables T_i are independent and identically distributed (equation (10)).

To be consistent with the fact that the fixed point lies at the root of its construction tree, we write $\nu_\emptyset$ for ν in (7) for the remainder of this paper.

4. Existence and uniqueness of a fixed point

This section makes precise the conditions under which the Galton–Watson IFS defined in equation (15) possesses a unique fixed point.

Theorem 4.1. *Let $(K, \mathcal{K}, \kappa)$ be the space of extended trees and define $\mathbb{L}_p$ using $(\Sigma, \mathcal{F}, P) = (K, \mathcal{K}, \kappa)$. If $\mathbb{E}_\kappa \sum_{j=1}^{\nu_\emptyset} r_j \int |\phi_j(0, x)|^p \, \mathrm{d}x < +\infty$ for some $1 < p < +\infty$, where r_j is the contractive factor of ϱ_j with $0 < r_j < 1$ almost surely, each $\phi_j(., .)$ is a.s. Lipschitz in its first variable, with Lipschitz constant S_j and $\lambda_p = \mathbb{E} \sum_{j=1}^{\nu_\emptyset} r_j S_j^p < 1$, where $\mathbb{E}$ denotes the expectation under κ, there exists a unique function f^* which satisfies $f^* = T f^*$ in $\mathbb{L}_p$. Moreover, for all $f_0 \in \mathbb{L}_p(\mathbb{X})$,*

$$d_p^*(T^n f_0, f^*) \leqslant \frac{\lambda_p^{n/p}}{1 - \lambda_p^{1/p}} d_p^*(f_0, T f_0) \tag{11}$$

which tends to 0 as $n \to +\infty$.

Proof. The proof is in two steps. We first need to check that $\mathbb{L}_p$ is closed under T. Next, we have to show that T is contractive in the complete metric space $(\mathbb{L}_p, d_p^*)$. The Banach fixed point theorem will ensure the existence and uniqueness of a limit function in $\mathbb{L}_p$.

Let $f \in \mathbb{L}_p$. We make explicit the construction of iid copies of $f \in \mathbb{L}_p$. Using notation of section 2.2, write f_k for the realization of f at point $k \in K$, then define $f_k^{(j)}$ by $f_k^{(j)} := f_{T_j(k)}$. Since the random variables T_j are iid, so are the functions $f_k^{(j)}$.

First step. Let $f \in \mathbb{L}_p$. We want to show that $Tf \in \mathbb{L}_p$, or equivalently $\mathbb{E} \int_{\mathbb{X}} |(Tf)(x)|^p \, \mathrm{d}x < +\infty$. To this end, first note that in the expression (7) of Tf, the indicator function partitions $\mathbb{X}$ into disjoint subintervals, so that the absolute value of the sum equals the sum of absolute values. Thus

$$\mathbb{E} \int_{\mathbb{X}} |(Tf)(x)|^p \, \mathrm{d}x = \mathbb{E} \sum_{j=1}^{\nu_\emptyset} \int_{\varrho_j(\mathbb{X})} \left| \phi_j \left[f^{(j)}\left(\varrho_j^{-1}(x)\right), \varrho_j^{-1}(x) \right) \right) \right|^p \, \mathrm{d}x.$$

Since ϱ_j are affine with contraction factor $0 < r_j < 1$, its inverse is also affine with almost everywhere existing Jacobian, and we can perform the change of variable $y = \varrho_j^{-1}(x)$. We bound the Jacobian of the transformation by r_j. $\mathbb{E} \int_{\mathbb{X}} |(Tf)(x)|^p \, \mathrm{d}x$ is therefore bounded above by

$$\mathbb{E} \sum_{j=1}^{\nu_\emptyset} r_j \mathbb{E} \left[\int_{\mathbb{X}} |\phi_j[f^{(j)}(y), y]|^p \, \mathrm{d}y | \phi_j \right]. \tag{12}$$

In (12), we have also used the law of total probability: $\mathbb{E}(\cdot) = \mathbb{E}[\mathbb{E}(\cdot \mid \{\nu_\emptyset, \{\phi_j, \varrho_j\}\})]$ where the second expectation is conditioned on the IFS parameters. Terms depending on ν and ϱ_j can be put outside the second expectation, leaving us with a term which only depends on ϕ_j, hence (12). Note that the term in the inner expectation does no longer depend on the ϱ_i's and is just $d_p^{*p}(\phi_j[f^{(j)}, Id], 0)$ after conditioning on the IFS parameters. *Id* stands for the

identity function and 0 is the zero function. Using the triangle inequality, and the fact that for any reals x and y we have $|x + y|^p \leqslant 2^p(|x|^p + |y|^p)$ it follows that $\mathbb{E} \int_{\mathbb{X}} |(Tf)(x)|^p \, dx$ is bounded by

$$2^p \mathbb{E} \sum_{j=1}^{\nu_\emptyset} r_j d_p^{*p}(\phi_j[f_\kappa^{(j)}, Id], \phi_j[0, Id]) + 2^p \mathbb{E} \sum_{j=1}^{\nu_\emptyset} r_j d_p^{*p}(\phi_j[0, Id], 0). \quad (13)$$

Using the Lipschitz property of the ϕ_j, the first term of (13) is smaller than

$$2^p \mathbb{E} \sum_{j=1}^{\nu_\emptyset} r_j S_j^p d_p^{*p}(f_\kappa^{(j)}, 0),$$

which is bounded since $f \in \mathbb{L}_p$. The second term of (13) is proportional to $\mathbb{E} \sum_{j=1}^{\nu_\emptyset} r_j \int |\phi_j(0, x)|^p \, dx$ and is finite by assumption. $\mathbb{E} \int_{\mathbb{X}} |(Tf)(x)|^p \, dx < +\infty$ follows.

Second step. We now prove the contractive property of T under the conditions of theorem 4.1. Take f and g in $\mathbb{L}_p$ and consider $d_p^{*p}(Tf, Tg)$. As in step 1, we expand expressions of Tf and Tg and swap the sum and absolute value, we then use the law of total probability and perform the change of variable $y = \varrho_j^{-1}(x)$, whose Jacobian is bounded by r_j. We obtain

$$d_p^{*p}(Tf, Tg) = \mathbb{E} \int |(Tf)(x) - (Tg)(x)|^p \, dx$$

$$\leqslant \mathbb{E} \sum_{j=1}^{\nu_\emptyset} r_j \mathbb{E}^* \left[\int_{\mathbb{X}} |\phi_j[f^{(j)}(y)), y] - \phi_j[g^{(j)}(y), y]|^p \, dy \right]$$

where $\mathbb{E}^* = \mathbb{E}[\cdot \mid \{\nu_\emptyset, \{\phi_i, \varrho_i\}\}]$. Lastly, we use the Lipschitz property of the nonlinear random maps ϕ_j to conclude that

$$d_p^{*p}(Tf, Tg) \leqslant \lambda_p d_p^{*p}(f, g),$$

where the definition of λ_p is given in the theorem statement. Under the assumption $\lambda_p < 1$, the contractive property follows and from the Banach fixed-point theorem there exists a unique function f^*, attractor of the Galton–Watson IFS. Moreover,

$$d_p^{*p}(T^n f_0, f^*) \leqslant \lambda_p d_p^{*p}(T^{n-1} f_0, f^*)$$

which leads to

$$d_p^*(T^n f_0, f^*) \leqslant \lambda_p^{n/p} d_p^*(f_0, f^*).$$

Now using the triangle inequality

$$d_p^*(f_0, f^*) \leqslant d_p^*(f_0, Tf_0) + \lambda_p^{1/p} d_p^*(f_0, f^*)$$

so that

$$d_p^*(T^n f_0, f^*) \leqslant \frac{\lambda_p^{n/p}}{1 - \lambda_p^{1/p}} d_p^*(f_0, Tf_0),$$

which concludes the proof of the theorem. $\qquad \square$

To illustrate, we present in figure 5 a realization of the fixed point of a certain IFS and its mean. The IFS parameters are detailed in the figure caption.

The theorem not only states that starting from an initial function the IFS converges in $\mathbb{L}_p$ to a unique fixed point under the metric d_p^* but also that the convergence is exponential. It follows that the convergence of $T^n f_0$ towards f^* is almost sure. To show this, let $\epsilon > 0$, then

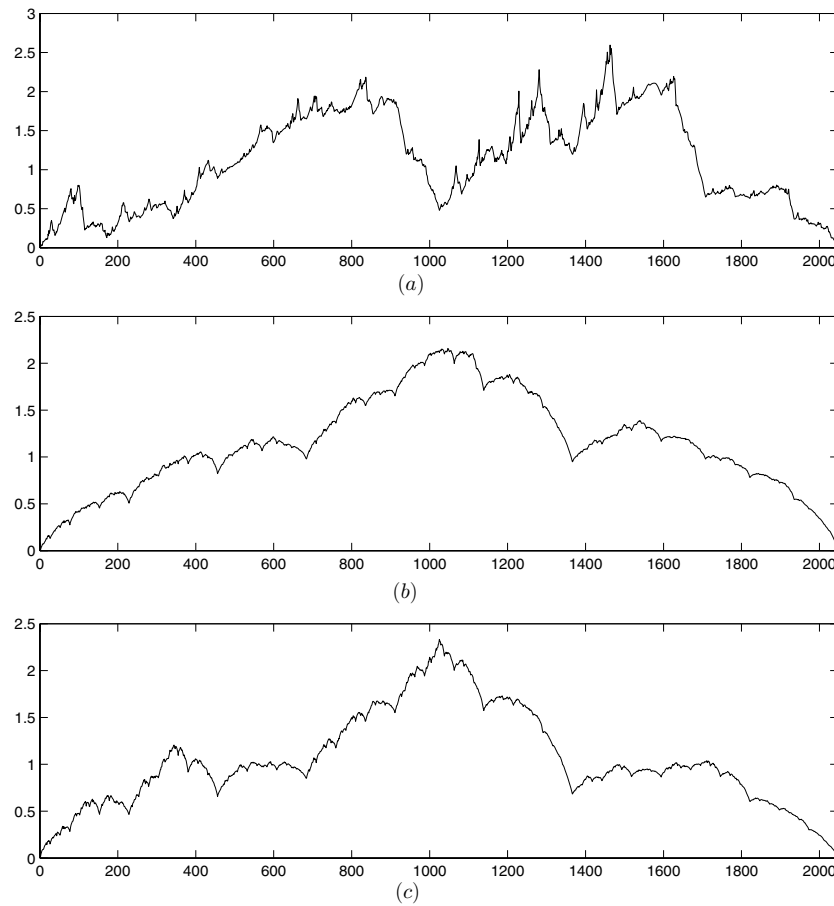$$P(d_p^p(T^n f_0, f^*) > \epsilon) \leqslant \frac{\mathbb{E} d_p^p(T^n f_0, f^*)}{\epsilon} \leqslant C \lambda_p^n,$$

Figure 5. A realization of the fixed point (*a*) and its mean (*b*). ϕ_j are decomposed as follows: $\phi_j(x,t) = s_j x + X \zeta_j(t)$ for $j = 1, \ldots, \nu$, where X is normally distributed with mean 1 and variance 0.25. When $\nu = 1, s_1 = 0.6$ and $\zeta_1(t) = t(1-t)$. For $\nu = 2$ we define $s_1 = 0.6$, $s_2 = 0.7, \zeta_1(t) = t^3, \zeta_2(t) = 1 - t^2$ and for $\nu = 3$ we have $s_1 = 0.6, s_2 = 0.7, s_3 = 0.3, \zeta_1(t) = t^4, \zeta_2(t) = (t+1)(1 - 0.75t^3)$ and $\zeta_3(t) = 0.5(1 - t^2)$. ν takes the values 1, 2 or 3 with probabilities 0.2, 0.3 and 0.5 for the first two figures. The bottom figure is the mean obtained with the probabilities 0.2, 0.2 and 0.6.

where

$$C = \frac{d_p^{*p}(f_0, T f_0)}{\epsilon \left(1 - \lambda_p^{1/p}\right)^p}.$$

It follows that

$$\sum_{n \geqslant 1} P\left(d_p^p(T^n f, f^*) > \epsilon\right) < \infty$$

and from the Borel–Cantelli lemma we have *P*-almost sure convergence.

f^* is the unique fixed point for which $f^* = T f^*$ in $\mathbb{L}_p$ but there may be some other $f^0 \neq f^*$ such that the law of f^0 equals the law of $T f^0$. The following result can be proven in the same way as Hutchinson and Rüschendorff [9].

Corollary 4.2. *The distribution of f^* is the unique distribution which satisfies $f^* \overset{d}{=} Tf^*$, where $\overset{d}{=}$ denotes equality in distribution.*

The idea is to define a new space of probability distributions of elements of $\mathbb{L}_p$ and a new metric over this space which lead to a complete metric space. Then one can prove that the operator T seen at the distribution level is contractive in this space and therefore admits a unique fixed point.

5. Properties of the fixed point

We now consider two properties of the fixed point. First, we derive conditions under which paths are a.s. continuous. Then, we look at the moments of the fixed point and show that under certain assumptions, moments of the attractor continuously depend on the probability vector **q**. This fact is suggested by observing figure 5 where a small change in **q** induces 'small' variations in the mean of the fixed point.

5.1. Continuity of the sample paths

The results for the Galton–Watson IFS are a straightforward generalization of continuity results in the deterministic setting.

Proposition 5.1. $\mathbb{X} = [a, b]$. *Let α be the unique random fixed point of $\phi_1(\cdot, a)$ and β the unique random fixed point of $\phi_{\nu_\emptyset}(\cdot, b)$: $\phi_1(\alpha, a) = \alpha$ and $\phi_{\nu_\emptyset}(\beta, b) = \beta$. Assume that α and β are the same for all possible realizations of ϕ_1 and $\phi_{\nu_\emptyset}$. If $\phi_i(\beta, b) = \phi_{i+1}(\alpha, a)$ a.s. for all $i \in \{1, \dots, \nu_\emptyset - 1\}$ and all the operators considered are continuous, then f^* has continuous paths and $f^*(a) = \alpha$ and $f^*(b) = \beta$ (a.s.).*

Proof. We first note that $f^*(a)$ and $f^*(b)$ are respectively fixed points of ϕ_1 and $\phi_{\nu_\emptyset}$:

$$f^*(a) = \phi_1\big[f^*\big(\varrho_1^{-1}(a)\big), \varrho_1^{-1}(a)\big] = \phi_1[f^*(a), a]$$
$$f^*(b) = \phi_{\nu_\emptyset}\big[f^*\big(\varrho_{\nu_\emptyset}^{-1}(b)\big), \varrho_{\nu_\emptyset}^{-1}(b)\big] = \phi_{\nu_\emptyset}[f^*(b), b].$$

Those equalities have to remain true whatever $\nu_\emptyset$ is, which is realized under the assumption of proposition 5.1.

We start the iteration with a continuous function f, and we consider Tf. Let $\varrho_i[a, b] = [a_{i-1}, a_i]$ for $i \in \{1, \dots, \nu_\emptyset\}$ and $a_0 = a, a_{\nu_\emptyset} = b$ almost surely. We only have to prove the continuity at the points a_i of the interval $[a, b]$ since it is the only place where Tf can be discontinuous, and d_p^* is complete on the set of continuous functions [9]. Therefore, if the nth iterate of T is continuous, the limit process also belongs to the space of continuous functions.

$f^*(a_i)$ can be expressed in two different ways as the point a_i is at the intersection of $\varrho_i[a, b]$ with $\varrho_{i+1}[a, b]$:

$$f^*(a_i) = \phi_i\big[f^*\big(\varrho_i^{-1}(a_i)\big), \varrho_i^{-1}(a_i)\big] = \phi_i[f^*(b), b] = \phi_i[\beta, b]. \tag{14}$$

We can show in a similar way that $f^*(a_i) = \phi_{i+1}[\alpha, a]$. Under the condition of the proposition the continuity of f^* at points a_i follows. $\qquad\square$

With this model, it is possible to obtain continuous paths or random processes everywhere discontinuous by adjusting the IFS parameters. Allowing only one discontinuity by not joining two operators $\phi_{\nu_\emptyset, i}$ and $\phi_{\nu_\emptyset, i+1}$ at the random point a_i will result in an everywhere discontinuous fixed point. A realization of a continuous fixed point is represented in figure 5.

5.2. Continuous dependency w.r.t. **q**

The continuity of the moments of the fixed point with respect to **q** is suggested in figure 5. This observation is related to that made by Barnsley in [10] where the attractor of a deterministic IFS is continuously varying with respect to the IFS parameters, leading to applications in image synthesis. We prove the result here for the model presented in [23], with deterministic maps and a random tree.

Consider the set of deterministic maps

$$\{\{\phi_{k,1}, \ldots, \phi_{k,k}, \varrho_{k,1}, \ldots, \varrho_{k,k}\}\}_{k=1,2,\ldots}.$$

Given $\nu_\emptyset = j$, then apply $\{\phi_{j,1}, \ldots, \phi_{j,j}, \varrho_{j,1}, \ldots, \varrho_{j,j}\}$. $\phi_{k,j}$ and $\varrho_{k,j}$ may differ for different values of $k, j = 1, \ldots, k$. The operator T becomes

$$(Tf)(x) = \sum_{j=1}^{\nu_\emptyset} \phi_{\nu_\emptyset,j}\big[f^{(j)}\big(\varrho_{\nu_\emptyset,j}^{-1}(x)\big), \varrho_{\nu_\emptyset,j}^{-1}(x)\big)\big]\mathbf{1}_{\varrho_{\nu_\emptyset,j}(\mathbb{X})}(x). \tag{15}$$

Lipschitz factor of $\phi_{\nu_\emptyset,j}$ is $S_{\nu_\emptyset,j}$ and the contraction factor of $\varrho_{\nu_\emptyset,j}$ is denoted by $r_{\nu_\emptyset,j}$. Since the operators attached to the branches of the tree are the same for a given number of offsprings, to one realization of the tree is associated one and only one realization of the fixed point.

Theorem 5.2. *Suppose conditions of theorem 4.1 hold. Let $f^* \in \mathbb{L}_p$ be the fixed point of a Galton–Watson IFS, bounded number of offspring and deterministic maps of the form $\phi(u, v) = su + \zeta(v)$, where $0 \leqslant s < 1$ and ζ is a nonlinear function. Suppose that $\lambda_r = \mathbb{E} \sum_{j=1}^{\nu} r_j S_j^r < 1$ for $r = 1, \ldots, p$. Then the rth moment of f^* continuously varies with respect to the probability generating vector **q**, for $r = 1, \ldots, p$.*

Proof. We prove the theorem by recurrence on r, for $r = 1, \ldots, p$. The first step of the proof shows that the continuity property holds for the mean of the fixed point. In the second step, we generalize it to any higher order integer moment. Let Υ be the space of probability vectors

$$\Upsilon = \left\{\mathbf{p} = (p_i, i \in \mathbb{N}^*)\,\middle|\, \sum_i p_i = 1\right\}$$

where $\mathbb{N}^* := \{1, 2, \ldots\}$. This space is endowed with the metric $l(\mathbf{p}, \mathbf{q}) = \sum_i |p_i - q_i|$.

First step. By definition of $\mathbb{L}_p$, $\mathbb{E}_\mathbf{p} f^* \in L_1$, if we denote by $f_\mathbf{p}^*$ the fixed point of the IFS with probability vector **p** and by $\mathbb{E}_\mathbf{p}$ the expectation under $\kappa_\mathbf{p}$, the probability measure defined on K with the probability generating vector **p**. We adopt this notation in this section to emphasize on **p**. Note that by changing the probability vector, we change the measure $\kappa_\mathbf{p}$ on the space of extended Galton–Watson trees. Therefore, if we now call $f_\mathbf{q}^*$ the fixed point of the same IFS with probability generating vector **q**, the expectation with respect to this new measure is different from $\mathbb{E}_\mathbf{p}$ and we denote it by $\mathbb{E}_\mathbf{q}$ (expectation under the new measure $\kappa_\mathbf{q}$). The continuity of the mean of the fixed point with respect to the generating vector follows if we show the continuity of the map $\psi : \Upsilon \to L_1$ which associates with each probability vector the mean of the fixed point of the Galton–Watson IFS. Let $\mathbf{p} \in \Upsilon$, we want to show that for all $\epsilon > 0$, there exists $\eta > 0$ such that

$$\forall \mathbf{q} \in \Upsilon \quad l(\mathbf{p}, \mathbf{q}) \leqslant \eta \Rightarrow d_1(\mathbb{E}_\mathbf{p} f_\mathbf{p}^*, \mathbb{E}_\mathbf{q} f_\mathbf{q}^*) \leqslant \epsilon. \tag{16}$$

Let $\epsilon > 0$ and $\mathbf{p} \in \Upsilon$. We first use the fact that f^* and Tf^* have the same distribution, therefore the same mean

$$
d_1(\mathbb{E}_{\mathbf{p}}f_{\mathbf{p}}^*, \mathbb{E}_{\mathbf{q}}f_{\mathbf{q}}^*) = \int \left| \mathbb{E}_{\mathbf{p}} \sum_{j=1}^{\nu_1} \phi_{\nu_1, j}\left[f_{\mathbf{p}}^* \circ \varrho_{\nu_1, j}^{-1}, \varrho_{\nu_1, j}^{-1}\right]\mathbf{1}_{\varrho_{\nu_1, j}(\mathbb{X})} \right.
$$
$$
\left. - \mathbb{E}_{\mathbf{q}} \sum_{j=1}^{\nu_2} \phi_{\nu_2, j}\left[f_{\mathbf{q}}^* \circ \varrho_{\nu_2, j}^{-1}, \varrho_{\nu_2, j}^{-1}\right]\mathbf{1}_{\varrho_{\nu_2, j}(\mathbb{X})} \right|,
$$

where $\kappa_{\mathbf{p}}(\nu_1 = k) = p_k$ and $\kappa_{\mathbf{q}}(\nu_2 = k) = q_k$. We omit the variable x in the integrand to keep the notation clear. By conditioning with respect to ν_1 and ν_2, the right-hand side becomes

$$
\int \left| \sum_{i \geqslant 1} p_i \sum_{j=1}^{i} \mathbb{E}_{\mathbf{p}} \phi_{i,j}\left[f_{\mathbf{p}}^* \circ \varrho_{i,j}^{-1}, \varrho_{i,j}^{-1}\right]\mathbf{1}_{\varrho_{i,j}(\mathbb{X})} - \sum_{i \geqslant 1} q_i \sum_{j=1}^{i} \mathbb{E}_{\mathbf{q}} \phi_{i,j}\left[f_{\mathbf{q}}^* \circ \varrho_{i,j}^{-1}, \varrho_{i,j}^{-1}\right]\mathbf{1}_{\varrho_{i,j}(\mathbb{X})} \right|.
$$

The sums can be taken outside the integral. By setting $y = \varrho_{i,j}^{-1}(x)$ and bounding the Jacobian by $r_{i,j}$, where $r_{i,j}$ is deterministic here as we consider non-random maps, $d_1(\mathbb{E}_{\mathbf{p}}f_{\mathbf{p}}^*, \mathbb{E}_{\mathbf{q}}f_{\mathbf{q}}^*)$ is less than

$$
\sum_{i,j} r_{i,j} \int |p_i \mathbb{E}_{\mathbf{p}} \phi_{i,j}[f_{\mathbf{p}}^*(y), y] - q_i \mathbb{E}_{\mathbf{q}} \phi_{i,j}[f_{\mathbf{q}}^*(y), y]| \, \mathrm{d}y.
$$

To go further, a particular form for the $\phi_{i,j}$ is required. We consider the case when they are pure contractions in their first variable and are nonlinear in their second variable: $\phi_{i,j}(u, v) = s_{i,j}u + \zeta_{i,j}(v)$. The Lipschitz factor of $\phi_{i,j}$ is $s_{i,j}$ in this case. Using the triangle inequality of $|\cdot|$ it follows that $d_1(\mathbb{E}_{\mathbf{p}}f_{\mathbf{p}}^*, \mathbb{E}_{\mathbf{q}}f_{\mathbf{q}}^*)$ is bounded by

$$
\sum_{i,j} r_{i,j} \left[\int s_{i,j} |p_i \mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^* - q_i \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^*| + |p_i - q_i \| \zeta_{i,j}(y)| \, \mathrm{d}y \right].
$$

The term $|p_i \mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^* - q_i \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^*|$ can be further bounded above by $|\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^* \| p_i - q_i| + q_i |\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^* - \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^*|$ by adding and subtracting $q_i \mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^*$ and using the triangle inequality. Suppose $\mathbf{p}$ and $\mathbf{q}$ are chosen such that $l(\mathbf{p}, \mathbf{q}) \leqslant \eta$. We have

$$
d_1(\mathbb{E}_{\mathbf{p}}f_{\mathbf{p}}^*, \mathbb{E}_{\mathbf{q}}f_{\mathbf{q}}^*) \leqslant \eta \sum_{i,j} r_{i,j} s_{i,j} \int |\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^*|
$$
$$
+ \sum_{i,j} q_i r_{i,j} s_{i,j} \int |\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^* - \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^*| + \eta \sum_{i,j} r_{i,j} \int |\zeta_{i,j}(y)| \, \mathrm{d}y.
$$

In the first term of the right-hand side, $\int |\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^*| < M < \infty$ since $\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^* \in L_1$. We have a bounded number of maps so $\sum_{i,j} r_{i,j} s_{i,j}$ is also bounded. In the second term, $\int |\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^* - \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^*|$ is the distance between $\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^*$ and $\mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^*$ and $\sum_{i,j} q_i r_{i,j} s_{i,j}$ is exactly the contraction factor $\lambda_1(\mathbf{q}) < 1$ of the map T. Since the map $\mathbf{p} \mapsto \sum p_i r_{i,j} s_{i,j}$ is linear, it follows that the distance between $\lambda_1(\mathbf{q})$ and $\lambda_1(\mathbf{p})$ is small for $\mathbf{p}$ sufficiently close to $\mathbf{q}$. Therefore $\lambda_1(\mathbf{q}) \leqslant 1 - \varepsilon_p$ for some small $\varepsilon_p > 0$. Finally, the third term is bounded by assumption. It follows that

$d_1(\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^*, \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^*)$ is smaller than

$$\frac{\eta}{\varepsilon_p} \left[M \sum_{i,j} r_{i,j} s_{i,j} + \sum_{i,j} r_{i,j} \int |\zeta_{i,j}(y)| \, \mathrm{d}y \right] := \frac{\eta}{\varepsilon_p} \gamma.$$

Set $\eta = \frac{\varepsilon_p}{\gamma} \epsilon$, it follows that $d_1(\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^*, \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^*) \leqslant \epsilon$.

Second step. We now show that the previous result holds for the *r*th moment of the fixed point, as long as $r \leqslant p$. To fix ideas, let us start with the second-order moment. First note that f^{*2} is distributed like $(Tf^*)^2$. Then, proceeding as before we bound $d_1(\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^{*2}, \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^{*2})$ by

$$\sum_{i,j} r_{i,j} \int \left| p_i \mathbb{E}_{\mathbf{p}} \phi_{i,j}^2 [f_{\mathbf{p}}^*(y), y] - q_i \mathbb{E}_{\mathbf{q}} \phi_{i,j}^2 [f_{\mathbf{q}}^*(y), y] \right| \mathrm{d}y.$$

By developing the square, the following terms appear: $s_{i,j}^2 f^{*2}$, $\zeta_{i,j}^2(y)$ and $2s_{i,j}\zeta_{i,j}(y)$. It follows that

$$d_1\left(\mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^{*2}, \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^{*2}\right) \leqslant \sum_{i,j} r_{i,j} \left[\int s_{i,j}^2 \left| p_i \mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^{*2} - q_i \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^{*2} \right| \right.$$

$$\left. + 2s_{i,j}\zeta_{i,j}(y) | p_i \mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^* - q_i \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^* | + \zeta_{i,j}^2(y) \, \mathrm{d}y \right].$$

Since for all $r \leqslant p$, $f \in \mathbb{L}_p(\mathbb{X}) \Rightarrow f \in \mathbb{L}_r(\mathbb{X})$, f^* is in $\mathbb{L}_2$ and the last term of the right-hand side is bounded above by assumption. The second term of the right-hand side is also smaller than η times some constant, as noted previously in step 1 of the proof. In the first term, we proceed as in step 1 and it follows that the right-hand side is less than some constant times η. The continuity follows.

When dealing with the *r*th order moment, the term $\phi_{i,j}^r [f^*(y), y]$ appears in the bounding factor. Expanding the expression, the terms $\int \left| p_i \mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^{*j} - q_i \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^{*j} \right|$ for $1 \leqslant j \leqslant r - 1$ show up, all less to some constant times η by the recurrence hypothesis. The conclusion follows by noting that the triangle inequality can be applied as before to the term $\int | p_i \mathbb{E}_{\mathbf{p}} f_{\mathbf{p}}^{*r} - q_i \mathbb{E}_{\mathbf{q}} f_{\mathbf{q}}^{*r} |$. $\qquad \square$

5.3. Test for multifractality

Fractal processes are by construction highly irregular. Information about the local fluctuations of a process $X(t)$ is made precise with the definition of the Hölder exponent at a specific time $t = t_0$. The process X is said to belong to $C_{t_0}^h$ if there is a polynomial P_{t_0} such that

$$|X(t) - P_{t_0}(t)| \leqslant K|t - t_0|^h$$

in a neighbourhood of t_0. The largest value H of h such that $X \in C_{t_0}^h$ is the Hölder exponent of X at $t = t_0$ [25]. Monofractal processes have a constant local Hölder exponent along their sample paths. In opposition, multifractals possess a richer structure. Their Hölder exponent behaves erratically with time: each interval of positive length exhibit a full range of different exponents. In practice, it is not possible to estimate the evolution of the Hölder exponent with time. Instead, multifractal processes are described by their Hausdorff spectrum $D(h)$, which gives the size (Hausdorff dimension) of sets with a given exponent h. The spectrum cannot be observed precisely in practice and alternative methods for its estimation have been proposed, giving birth to the multifractal formalism. $D(h)$ is usually estimated via the Legendre transform of a partition function $\zeta(q)$, obtained as a power law behaviour of multiresolution quantities. Jaffard, Lashermes and Abry have proposed an estimator of $\zeta(q)$
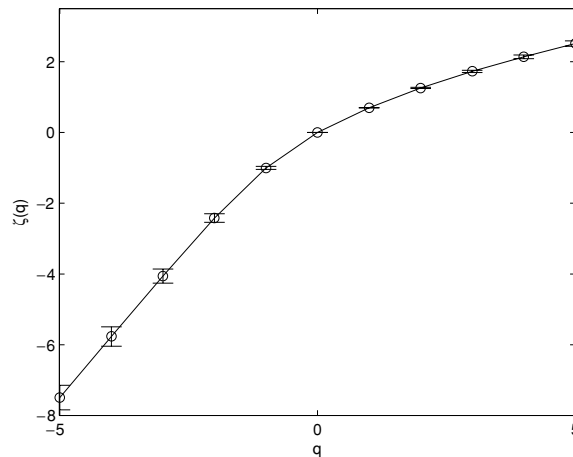
14

Figure 6. Estimation of the partition function $\zeta(q)$ of the fixed point of a Galton–Watson IFS using wavelet leaders. The parameters of the IFS are given in the figure caption 5, with ν taking values 1, 2 or 3 with probabilities 0.2, 0.3 and 0.5 respectively. The partition function is obtained by averaging 100 estimations. 95% confidence intervals are also plotted.

using wavelet leaders [26]. The Legendre transform of $\zeta(q)$ provides in general an upper bound of the Hausdorff spectrum of X

$$D(h) \leqslant \inf_{q \neq 0}(1 + qh - \zeta(q)).$$

Consider the polynomial expansion of $\zeta(q) = c_1 q + c_2 q^2/2 + c_3 q^3/6 + \cdots$. When $c_p = 0$ for all $p \geqslant 2$, $\zeta(q)$ is linear with q and the Hausdorff spectrum degenerates to a single point: the process is monofractal. Any departure from a linear behaviour is characteristic of multifractals. Wendt and Abry have designed a test to decide whether $c_p = 0$ or not [27]. In particular, the case $p = 2$ permits to conclude between a mono and multifractal process. The null hypothesis $c_p = c_{p,0}$ (H_0) is tested versus the two sided alternative $c_p \neq c_{p,0}$. The test statistic is therefore

$$T = c_p - c_{p,0}.$$

The distribution of the test statistic T under the null hypothesis is unknown in general and is estimated using non-parametric bootstrap techniques. From the empirical distribution, one can design an acceptance region $\mathcal{T}_{1-\alpha} = [t_{\alpha/2}, t_{1-\alpha/2}]$ where t_α is the α quantile of the null distribution. In the present study, we set $\alpha = 0.1$ and $c_{p,0} = 0$. We refer the reader to [27] for further details about the test statistics.

Consider the Galton–Watson IFS presented in figure 5 with ν taking values 1, 2 or 3 with probabilities 0.2, 0.3 and 0.5. We simulate 100 realizations of the fixed point, each of length 2^{12} and estimate the partition function using wavelet leaders. The wavelet coefficients are computed with Daubechies wavelets with two vanishing moments and the scale of analysis ranges from $j_1 = 3$ to $j_2 = 12$. The partition function presented in figure 6 is obtained by averaging 100 estimations of $\zeta(q)$. It clearly appears nonlinear, which suggests the multifractal behaviour of the fixed point. Also, in the multifractality test previously described, H_0 is rejected 99% of time for $p = 2$, confirming nonlinear shape of $\zeta(q)$. Those observations indicate the existence of a class of Galton–Watson fixed points which are multifractals. This result motivates a further study in which one could derive conditions on the IFS parameters in order to obtain multifractal fixed points.

6. Conclusion

This study extends the model proposed by Hutchinson and Rüschendorff by adding more randomness to it, considering random operators on a random underlying construction tree. The strict self-similarity observed in attractors of deterministic IFS is a drawback when modelling natural signals. Adding randomness gives visually more interesting models and it would be interesting to consider parameter estimation problems on Galton–Watson IFS. To start with, one could give operators a particular form and estimate the offspring distribution of the random tree.

Further theoretical properties of the fixed point also need to be carried out, starting with its multifractal analysis. The cascade nature of IFS lets us imagine a non-trivial multifractal spectrum of the fixed point, intuition confirmed by the simulations performed in the previous section. Existing results for deterministic IFS on functions [8] and for random IFS on measures [28] motivate this study.

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